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Turning the resistive MHD into a stochastic field theory

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Abstract. Classical systems stirred by random forces of given statistics may be described via a path integral formulation in which their degrees of freedom are stochastic variables themselves, undergoing a multiple-history probabilistic evolution. This framework seems to be easily applicable to resistive Magneto-Hydro-Dynamics (MHD). Indeed, MHD equations form a dynamic system of classical variables in which the terms representing the density, the pressure and the conductivity of the plasma are irregular functions of space and time when turbulence occurs. By treating those irregular terms as random stirring forces, it is possible to introduce a Stochastic Field Theory which should represent correctly the impulsive phenomena caused by the spece- and time-irregularity of plasma turbulence. This work is motivated by the recent observational evidences of the crucial role played by stochastic fluctuations in space plasmas.

1 Introduction

The study of space plasmas is probably one of the richest branches of non-quantum physics in terms of specific theoretical tools to be invoked, due to the intrinsic phenomenological variety of the systems studied.

The traditional approach to space plasma phenomena is based on Magneto-Hydro-Dynamics (MHD) in which plasma media are considered as smooth "deterministic" continua. In such a framework, the evolution of space plasmas in the presence of magnetic fields is described by functions of space and time, which are differentiable almost everywhere. In this representation, several processes can be treated by involving the evolution of simple plane waves, at least locally. In spite of the inherent simplification, the MHD description of space plasma processes has encoutered a wide success, especially in describing the large scale evolution (Kallenrode, 2001). This is particularly true in treating solar and magnetospheric phenomena, as well as interplanetary plasma dynamics (Choudury, 1998).

The last decade of the XX Century has seen a substantial change in the way of studying the space plasma phenomena. For example, in the framework of the magnetospheric processes, some studies pointed out that both the global, large scale dynamics of some magnetospheric regions (plasma sheet and central plasma sheet regions) and some internal processes related to magnetotail plasma transport could be better explained in terms of stochastic processes, low-dimensional chaos, fractal features, intermittent turbulence, complexity and criticality (see e.g. Chang (1992); Klimas et al. (1996); Chang (1999); Consolini (2002); Uritsky et al. (2002); Zelenyi and Milovanov (2004) and references therein). More in general, it becomes evident that MHD turbulence admitting singularities and stochastic MHD processes play a crucial role in several solar system plasma contexts, as for instance in the framework of interplanetary solar wind (Bruno and Carbone, 2005).

As far as stochasticity is concerned, let us remember that recently, Lazarian et al. (2004) have considered stochastic reconnection in a magnetized, partially ionized medium. Here, stochasticity arises from field line wandering through the turbulent fluid. Their results show an improvement in the calculation of the reconnection rate with respect to precedent "deterministic" schemes. A more general result in this framework has been achieved by ourselves (Materassi and Consolini, 2007) by considering the diffusion region as a fractal domain (a non-space filling region). Furthermore, in a different context Consolini et al. (2005) showed that stochastic fluctuations play a crucial role in a magnetospheric process, the tail current-disruption, occurring at the substorm onset.

Another relevant feature of space plasma is the nearly overall emergence of a non-Gaussian statistics of the small-scale magnetic field and plasma parameter fluctuations/increments. This feature observed in several different

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contexts, from solar wind to geotail plasma sheet, is generally associated with intermittency, i.e. the inhomogeneity of the scaling features at the small scales. On the other hand, recent 2-D MHD simulations (Wu and Chang, 2000; Chang et al., 2004) evidenced how the presence of spontaneous or forced fluctuations naturally generates multiscale coherent plasma and/or magnetic field structures, which can be considered as stochastic bundles of non-propagating fluctuations (Chang, 1999). The non-Gaussian features of small scale fluctuations has been related to the presence of such coherent multiscale structures (Chang et al., 2004). Coherent structures have been observed in several space plasma regions: in solar wind (see e.g. Bruno et al., 2001) as field-aligned flux tubes, in the Earth's cusp regions (see e.g. Yordanova et al., 2005), in the geotail plasma sheet (see e.g. Milovanov et al., 2001; Borovsky and Funsten, 2003; Vörös et al., 2004; Kretzschmar and Consolini, 2006) as current structures, 2-D eddies and so on. According to our opinion, recent observations of small-scale magnetic .eld features in the magnetosheath transition region and dissipation structures (Retinò et al., 2007; Sundkvist et al., 2007b) suggest that the dynamics of small scale structures can be the origin of a coherent dissipation mechanism, a sort of coarse-grained dissipation (Tetrault, 1992a,b) due to non-local interactions that result in the k-space.

All the aforementioned theoretical and experimental arguments point to the emergence of a complex dynamics due to the stochastic evolution of coherent structures, as well as to the crucial role of spontaneous and/or forced MHD fluctuations in irreversible and fast relaxation processes. In other words, the dynamics of many space plasmas could be similar to that of stochastic multiscale granular systems. MHD numerical simulations substantiate and support this scenario (Chang et al., 2004)

Consider then the role of singularity, as something normal for the physics of these systems, due for instance to impulsive and irreversible fast relaxation processes (such as reconnection): one can then notice that the MHD smooth scheme encounters some problems in coping with such singular phenomena, in which local topological *sudden changes* are observed. In some sense, these phenomena resemble more closely a quantum transition than a classical evolution. Indeed, istantaneous "classical" configurations of turbulent plasmas should be thought of as non-differentiable quantities, at least within some interval of space- and time-scale (Kallenrode, 2001; Yordanova et al., 2004).

Localized occurrence of big fluctuations in the medium (e.g. the resistivity in the case of reconnection) probably initiate and determine those *quantum-like* transitions of the MHD variables: if those fluctuations are treated as probabilistic stirring forces a range of possibilities to explain consistently the *sudden changes* appears, much more than in the traditional equilibrium scheme. Indeed, if the local thermal equilibrium is assumed (Kelley, 1989), only quasi-static thermodynamical changes are permitted, and all the non-quasi-static highly dissipative transformations that could allow for (topological) sudden transitions in the MHD variables are excluded unless we do not include instability sources, i.e. we do not go beyond the assumption of local equilibrium, and include the nonlinearities.

The presence of stochastic stirring forces makes the continuum obey proper *Langévin Equations*, yielding a collection of different evolutions starting from a fixed set of initial conditions, each evolution corresponding to a particular realization of the stochastic terms (Haken, 1983): the description of such systems may be given rather transparently in terms of path integrals (Feynman and Hibbs, 1965).

In this paper the MHD equations are re-interpreted as Langévin field equations. Then, following the brilliant trace of Phythian (1977), a proper Stochastic Field Theory (SFT) is defined for them, where the field variables have a probabilistic evolution, described via path integrals. Path integrals have already been used in specific problems of the space plasma physics: in Crew and Chang (1988) the probabilistic behaviour of the kinetic levels of the theory is directly represented via Boltzmann's distributions. The present paper rather deals with the formulation of a plasma continuum, single fluid SFT and its motivations.

The path integral representation is also very suitable to investigate multi-scale dynamical aspects, because the technique of *Renormalization Group* is naturally applied to this representation (see e.g. Chang et al. (1992) and references therein). For instance, the first direct application of such technique, using the exact full dynamic differential renormalization group for critical dynamics can be found in (Chang et al., 1978). The use of Renormalization Group techniques to predict physical quantities to be compared with real spacecraft data is already well established (see e.g. Chang, 1999; Chang et al., 2004), and the results are very encouraging.

Before proceeding to the formulation of a SFT for the MHD it is useful to stress that the major target of this work is to introduce a stochastic Lagrangian scheme able to describe the evolution of MHD systems in the presence of stochastic fluctuations, and to discuss the physical reasoning for the choice of stochastic elements (stirring forces). In passing we remark that practical applications of the scheme here presented to specific physical cases go beyond the aim of this work and are delayed to forthcoming papers.

The paper is organized as follows. In Sect. 2 the framework given in Phythian (1977) is briefly reviewed. Section 3 is the core of the results presented here: all the terms necessary to construct the MHD stochastic field theory are written explicitly. Section 4 deals with the problem of using the statistical knowledge of the irregular plasma in building up the SFT concretely for the resistive MHD: as a toy model, the example in which the stirring force statistics is directly assumed to be Gaussian, is presented. Section 5 finally points out the main developments expected from this work, and those questions left open in it.

2 Phythian's formalism

Consider a classical system described by a set of field variables ψ undergoing a *Langévin equation* of motion

$$\dot{\psi}^I(\boldsymbol{x},\tau) = \Lambda^I[\psi;\boldsymbol{x},\tau] + \int g_J(\boldsymbol{y},\tau)\,\Gamma^{JI}[\psi;\tau,\boldsymbol{y},\boldsymbol{x}]\,d^3y + f^I(\boldsymbol{x},\tau),$$

where f and g are stirring forces governed by given statistics, while Λ and Γ are "deterministic" expressions. Square brackets like in $\Lambda^I[\psi;\boldsymbol{x},\tau]$ and $\Gamma^{JI}[\psi;\tau,\boldsymbol{y},\boldsymbol{x}]$ underline functional, possibly non-local, dependences. Summation over repeated indices in contravariant positions is intended.

The presence of the probabilistic terms f and g makes the system evolve probabilistically along all the histories $\psi(\boldsymbol{x},\tau)$ for $\tau\in[t_0,t]$. In Phythian (1977) the statistical dynamics of such a system is turned into a *path integral formalism* with a "many history representation" of its evolution. Each history $\tilde{\psi}(\boldsymbol{x},\tau)$ is weighted with the *probability density* $\mathcal{A}\left[\tilde{\psi};t_0,t\right)$ that at the time $\tau\in[t_0,t]$ the point $\psi(\boldsymbol{x},\tau)$ belongs to a small neighborhood of $\tilde{\psi}(\boldsymbol{x},\tau)$. The kernel $\mathcal{A}$ is constructed as:

$$\mathcal{A}[\psi;t_0,t) = \int[d\chi]\,A[\psi,\chi;t_0,t), \tag{1}$$

where the auxiliary kernel A is defined in the following way:

$$\begin{cases} A[\psi,\chi;t_0,t) = \\ = N_0(t_0,t)\,C[\chi,\Gamma;t_0,t)\,e^{-i\int_{t_0}^{t}d\tau\int d^3x\left[\dot{\psi}^I(\boldsymbol{x},\tau)\chi_I(\boldsymbol{x},\tau)-\Lambda^I[\psi;\boldsymbol{x},\tau)\chi_I(\boldsymbol{x},\tau)-\frac{i}{2}\frac{\delta\Lambda^I[\psi;\boldsymbol{x},\tau)}{\delta\psi^I(\boldsymbol{x},\tau)}\right]}, \\ C[\chi,\Gamma;t_0,t) = \\ = \left\langle e^{i\int_{t_0}^{t}d\tau\int d^3x\left[f^I(\boldsymbol{x},\tau)\chi_I(\boldsymbol{x},\tau)+\int d^3y\,g_I(\boldsymbol{x},\tau)\chi_J(\boldsymbol{y},\tau)\Gamma^{IJ}[\psi;t,\boldsymbol{y},\boldsymbol{x})+g_I(\boldsymbol{x},\tau)\int d^3y\frac{\delta\Gamma^{IJ}[\psi;\tau,\boldsymbol{x},\boldsymbol{y})}{\delta\psi^J(\boldsymbol{y},\tau)}\right]}\right\rangle_{f,g}. \end{cases} \tag{2}$$

The last term $\frac{\delta\Lambda^I}{\delta\Psi^I}$in the exponential defining $A[\Psi;\chi;t_0,t)$ looks like a diver- gence in the functional space (T. Chang, personal communication, 2005), and indeed it is a *curvature term* that will not aect the dynamics, at least in the case of the MHD.

The coefficient $N_0(t_0,t)$ is a normalization factor, since $A[\psi,\chi;t_0,t)$ must be a probability density properly normalized:

$$\int[d\psi]\int[d\chi]\,A[\psi,\chi;t_0,t) = 1. \tag{3}$$

The variables χ, referred to as *stochastic momenta*, are as many as the physical variables ψ. They are introduced in order to render self-consistent mathematically this construction, and will exit the play at a physical level.

The operation of going from $A[\psi,\chi;t_0,t)$ to $\mathcal{A}[\psi;t_0,t)$ is not trivial in general, and its feasibility will depend on the feasibility of the calculation of $C[\chi,\Gamma]$. The form of the quantity to be integrated in C is relevant, as discussed in Feynman and Hibbs (1965).

The *ensemble average over all the possible histories from* t_0 *to* t *of* χ *and* ψ of any quantity $\mathcal{F}[\psi,\chi]$ is defined as

$$\langle\mathcal{F}\rangle = \int[d\psi]\int[d\chi]\,A[\psi,\chi;t_0,t)\,\mathcal{F}[\psi,\chi]. \tag{4}$$

This is the starting point to build up the SFT.

The *evolution probability from an initial field configuration to a final one*, both fully assigned as

$$\psi(t_0) = \psi^{(i)},\ \psi(t) = \psi^{(f)}, \tag{5}$$

is indicated as $\mathcal{P}_{\psi^{(i)}\to\psi^{(f)}}(t_0,t)$. It can be obtained simply by fixing the initial and final conditions and putting $\mathcal{F}$=1 in the calculation of $\langle\mathcal{F}\rangle$ (Polchinski, 1994):

$$\mathcal{P}_{\psi^{(i)}\to\psi^{(f)}}(t_0,t) = \langle 1\rangle_{\substack{\psi(t_0)=\psi^{(i)}\\ \psi(t)=\psi^{(f)}}} = \int_{\substack{\psi(t_0)=\psi^{(i)}\\ \psi(t)=\psi^{(f)}}}[d\psi]\int[d\chi]\,A[\psi,\chi;t_0,t). \tag{6}$$

From the point of view of complexity, the formulation just reported is very interesting because it contains elements of non-polinomiality and non-locality, as soon as the noise terms do. Let's indeed consider the definition of a *stochastic action* S so that

$$A = N_0 \exp(-iS), \tag{7}$$

that includes the term

$$S_C[\chi, \Gamma] = i \ln C[\chi, \Gamma] \tag{8}$$

(see Eq. 2), in which all the noise properties are encoded: due to the presence of $S_C[\chi, \Gamma]$, the resulting theory is *non-polynomial*, since in general there is no argument to truncate the expansion in multi-point and multi-time correlation functions of f and g defining this addendum (Chang, 1999). By the way, even the multiscale nontrivial features of the noise enter the theory here.

The properties of non-locality and non-polynomiality inherited from the noise correlation terms yield the necessity of finding some more handleable formalism, and this suggests to work with the *Renormalization Group* (RG). Indeed, renormalization of a theory may lead to so deep changes in its coefficients to convert polynomial theories into non-polynomial ones, local theories into non-local ones, and viceversa (Ma, 1973). However, the actual necessity and opportunity of employing a RG framework or not, must be suggested by the physics of the system at hand: in space plasmas there is the experimental evidence that multi-scale dynamics appears, and many (geo)space systems may well be considered statistical systems near criticality (Chang, 1992).

3 Application to the resistive MHD

The results of Section 2 may be applied to the resistive MHD theory equations. For a locally neutral plasma the resistive MHD equations written in a vector component form read

$$\begin{cases} \partial_t B^i = B^j \partial_j V^i - B^i \partial_j V^j - V^j \partial_j B^i - \epsilon^{ijk} \partial_j \left(\zeta_{kh} J^h\right), \\ \partial_t V^i = V^i \partial_j V^j - V^j \partial_j V^i + \dfrac{J_j}{\rho} B_k \epsilon^{jki} - \dfrac{\partial^i p}{\rho}, \end{cases} \tag{9}$$

where

$$\zeta = \sigma^{-1} \tag{10}$$

is the *resistivity tensor*, ρ is the *mass density* of the plasma and p is its *pressure*. The dynamical variables are the plasma velocity $\boldsymbol{V}$ and the magnetic field $\boldsymbol{B}$.

The form of the quantities ζ, ρ and p, and the mathematical relationships among themselves, depend on the microscopic nature of the medium. In the traditional fluid-dynamical scheme (Materassi, 2002), *constitutive hypotheses* provide the information on the microscopic nature of the medium. One assumes the (at least local) *thermodynamical equilibrium*, so that the constitutive hypotheses read

$$\zeta = \zeta(T, ...), \; \Phi(\rho, p, T) = 0, \tag{11}$$

being T the temperature field. One shall then invoke some *heat conduction equation*, requiring other constitutive hypotheses about the specific heats of the plasma. This produces a temperature field equation closing the system.

Turbulent plasmas can instead be considered as *out-of-equilibrium systems* (Treumann, 1998, 1999a,b; Consolini et al., 2006). ζ, ρ and p may be *very irregular functions of $\boldsymbol{x}$ and t*, with high variability on distances and times much smaller than the MHD scale.

According to Treumann (1999a,b) one could use the traditional scheme, modified to take into account of the non-Gaussianity where the conditions Eq. (11) are obtained in the framework of the Lorentzian thermodynamics of the turbulent κ gases, a very promising construction.

Otherwise, irregularities in ζ, ρ and p may be explicitly considered by stating that ζ, ρ and p are *stochastic fields* and assigning their *probability density functions* (PDFs). Then the randomness of the terms ζ, ρ and p may be transferred to the dynamical variables $\boldsymbol{B}$ and $\boldsymbol{V}$ via Phythian's scheme (Phythian, 1977), and the SFT may be constructed. This is the basic assumption chosen here.

If the following vector quantities are defined

$$\Xi^i = -\epsilon^{ijk}\partial_j\left(\zeta_{kh}J^h\right),\ \Delta^i = \frac{J^i}{\rho},\ \Theta^i = -\frac{\partial^i p}{\rho} \tag{12}$$

and considered as *random stirring forces* with known probability density functional $\mathcal{Q}[\boldsymbol{\Xi}, \boldsymbol{\Delta}, \boldsymbol{\Theta}]$, the *Langévin equations for the resistive MHD* will be written as

$$\begin{cases} \partial_t B^i = B^j\partial_j V^i - B^i\partial_j V^j - V^j\partial_j B^i + \Xi^i, \\ \partial_t V^i = V^i\partial_j V^j - V^j\partial_j V^i + \Delta_j B_k\epsilon^{jki} + \Theta^i. \end{cases} \tag{13}$$

The positions Eq. (12) and their consequence Eq. (13) are very possibly not the only way of turning the MHD into a Langévin theory. They rather have the advantage of reproducing exactly the Langévin equations on which the framework in Phythian (1977) is based.

Now, the Eq. (13) for the resistive MHD may be turned into a SFT by identifying ψ as $\boldsymbol{B}$ and $\boldsymbol{V}$, and defining as many stochastic momenta χ as the six ψs:

$$\psi = \boldsymbol{B}\oplus\boldsymbol{V},\ \chi = \boldsymbol{\Omega}\oplus\boldsymbol{\Pi}.$$

The stochastic kernel $A[\boldsymbol{\Omega}, \boldsymbol{\Pi}, \boldsymbol{B}, \boldsymbol{V}; t_0, t)$ will be constructed by involving a stirring force factor

$$C[\boldsymbol{\Omega}, \boldsymbol{\Pi}, \boldsymbol{B}, \boldsymbol{V}; t_0, t) = = \left\langle e^{i\int_{t_0}^t d\tau\int d^3x\left[\Xi^i(\boldsymbol{x},\tau)\Omega_i(\boldsymbol{x},\tau)+ +\Theta^i(\boldsymbol{x},\tau)\Pi_i(\boldsymbol{x},\tau)+\epsilon^{ijk}\Delta_i(\boldsymbol{x},\tau)\Pi_j(\boldsymbol{x},\tau)B_k(\boldsymbol{x},\tau)\right]}\right\rangle_{\boldsymbol{\Xi},\boldsymbol{\Delta},\boldsymbol{\Theta}} : \tag{14}$$

all the statistical dynamics of the resistive MHD interpreted as a stochastic field theory is then encoded in the kernel

$$A[\boldsymbol{\Omega}, \boldsymbol{\Pi}, \boldsymbol{B}, \boldsymbol{V}; t_0, t) = N_0(t_0, t)\, C[\boldsymbol{\Omega}, \boldsymbol{\Pi}, \boldsymbol{B}, \boldsymbol{V}; t_0, t) \; e^{-i\int_{t_0}^t d\tau\int d^3x\left(\Omega_i\dot{B}^i+\Pi_i\dot{V}^i+\left(B^i\partial_jV^j+V^j\partial_jB^i-B^j\partial_jV^i\right)\Omega_i+\left(V^j\partial_jV^i-V^i\partial_jV^j\right)\Pi_i\right)}. \tag{15}$$

No explicit form for $A[\boldsymbol{\Omega}, \boldsymbol{\Pi}, \boldsymbol{B}, \boldsymbol{V}; t_0, t)$ can be given until the explicit expression of $C[\boldsymbol{\Omega}, \boldsymbol{\Pi}, \boldsymbol{B}, \boldsymbol{V}; t_0, t)$ is found by making the necessary integrations on the stirring force terms. Further, one obtains the kernel $\mathcal{A}$ involving only physical fields

$$\mathcal{A}[\boldsymbol{B}, \boldsymbol{V}; t_0, t) = \int[d\boldsymbol{\Omega}]\int[d\boldsymbol{\Pi}]\, A[\boldsymbol{\Omega}, \boldsymbol{\Pi}, \boldsymbol{B}, \boldsymbol{V}; t_0, t) \tag{16}$$

once the integration over $\boldsymbol{\Omega}$ and $\boldsymbol{\Pi}$ is feasible.

The plasma physics will enter the present framework through the dynamical PDF

$$\mathcal{P}_{\text{dyn}} = \mathcal{P}_{\text{dyn}}[\zeta, \rho, p] : \tag{17}$$

as far as $\mathcal{P}_{\text{dyn}}[\zeta, \rho, p]$ keeps trace of the plasma complex dynamics, this represents a (rather general) way to provide *constitutive hypotheses* on the medium. The logical path hence is:

$$\text{(complex) particle} - \text{field dynamics} \longmapsto \mathcal{P}_{\text{dyn}}[\zeta, \rho, p]. \tag{18}$$

Then, the positions Eq. (12) are used to construct mathematically the passage

$$\mathcal{P}_{\text{dyn}}[\zeta, \rho, p] \longmapsto \mathcal{Q}[\boldsymbol{\Xi}, \boldsymbol{\Delta}, \boldsymbol{\Theta}]. \tag{19}$$

The form of $\mathcal{Q}[\boldsymbol{\Xi}, \boldsymbol{\Delta}, \boldsymbol{\Theta}]$ is clearly related to the dynamics of the microscopic degrees of freedom of the plasma. A closed form for $\mathcal{Q}[\boldsymbol{\Xi}, \boldsymbol{\Delta}, \boldsymbol{\Theta}]$ should be obtained consistently with *any* microscopic dynamical theory of the turbulent plasma, from the very traditional equilibrium statistical mechanics to the fractional kinetics reviewed in Zaslavsky (2002).

The calculation of impulsive processes in which suddenly the magnetized plasma changes arbitrarily, from an initial configuration $(\boldsymbol{B}(t_0), \boldsymbol{V}(t_0)) = (\boldsymbol{B}_\text{i}, \boldsymbol{V}_\text{i})$ to a final one $(\boldsymbol{B}(t), \boldsymbol{V}(t)) = (\boldsymbol{B}_\text{f}, \boldsymbol{V}_\text{f})$ may be done: the rate of such stochastic transitions should be calculated from

$$\mathcal{P}_{(\boldsymbol{B}_\text{i},\boldsymbol{V}_\text{i})\to(\boldsymbol{B}_\text{f},\boldsymbol{V}_\text{f})}(t_0, t) = \int_{\substack{(\boldsymbol{B}(t_0),\boldsymbol{V}(t_0))=(\boldsymbol{B}_\text{i},\boldsymbol{V}_\text{i}) \\ (\boldsymbol{B}(t),\boldsymbol{V}(t))=(\boldsymbol{B}_\text{f},\boldsymbol{V}_\text{f})}} [dB][dV]\,\mathcal{A}[\boldsymbol{B}, \boldsymbol{V}; t_0, t). \tag{20}$$

An entire representation *à la Feynman* of such processes is to be derived from the SFT.

In order to arrive to a closed expression for some stochastic action at least in one example case, in the next Section a toy model is defined, in which $\mathbf{\Xi}$, $\mathbf{\Delta}$ and $\mathbf{\Theta}$ are assumed to be Gaussian processes without any memory. This hypothesis is surely over-simplifying for a space plasma, since there are experimental results stating the presence of non-Gaussian distributions (Yordanova et al., 2005), and also of memory effects (Consolini et al., 2005). Nevertheless, the Gaussian example is of some use in illustrating the framework.

4 The Gaussian toy model

This toy model is worth being studied, because in this case the calculation of a Lagrangian defining locally the stochastic action can be performed analytically to the end. Moreover, the idea that $\mathbf{\Xi}$, $\mathbf{\Delta}$ and $\mathbf{\Theta}$ are determined by the concurrence of a large number of microscopic processes converging to a Gaussian statistics makes some sense (Yanovsky et al., 2000).

Let us assume $\mathbf{\Xi}$, $\mathbf{\Delta}$ and $\mathbf{\Theta}$ to be Gaussian processes fluctuating around some classical configuration $\mathbf{\Xi}_0(\boldsymbol{x},\tau)$, $\mathbf{\Delta}_0(\boldsymbol{x},\tau)$ and $\mathbf{\Theta}_0(\boldsymbol{x},\tau)$. The PDF of the configuration of the stirring forces at the time τ is assumed to depend only on the time considered: in other words, the *no memory effect hypothesis* is done. The stirring force $\mathbf{\Xi}(\boldsymbol{x},\tau)$ has (local) probability density function reading:

$$Q_{\Xi}(\mathbf{\Xi}(\boldsymbol{x},\tau)) = \sqrt{\frac{\det\left\|a_{\Xi}^{ij}(\boldsymbol{x},\tau)\right\|}{\pi^3}}\, e^{-a_{\Xi}^{ij}(\boldsymbol{x},\tau)\left[\Xi_i(\boldsymbol{x},\tau)-\Xi_{0i}(\boldsymbol{x},\tau)\right]\left[\Xi_j(\boldsymbol{x},\tau)-\Xi_{0j}(\boldsymbol{x},\tau)\right]}. \tag{21}$$

The matrix $a_{\Xi}^{ij}(\boldsymbol{x},\tau)$ is assumed to be symmetric, non-singular and positive definite, hence it may be written as diagonal:

$$a_{\Xi}^{ij} = \lambda_{\Xi}^{(i)}\delta^{ij}. \tag{22}$$

As far as the three eigenvalues $\lambda_{\Xi}^{(i)}$ were taken to be different, the statistics of $\mathbf{\Xi}$ should be considered anisotropic for some “intrinsic microscopic” reason. At this stage, a_{Ξ}^{ij}, a_{Δ}^{ij} and a_{Θ}^{ij} will be all taken as isotropic:

$$a_{\Xi}^{ij} = a_{\Xi}\delta^{ij},\ a_{\Delta}^{ij} = a_{\Delta}\delta^{ij},\ a_{\Theta}^{ij} = a_{\Theta}\delta^{ij}. \tag{23}$$

From Eq. (21) and analogous expressions for $\mathbf{\Delta}$ and $\mathbf{\Theta}$, it is possible to give the C pre-factor as

$$\begin{aligned} &C[\mathbf{\Omega},\mathbf{\Pi},\boldsymbol{B},\boldsymbol{V};t_0,t) = e^{-i\int_{t_0}^{t}d\tau L_C[\mathbf{\Omega},\mathbf{\Pi},\boldsymbol{B},\boldsymbol{V};\tau)},\\ &L_C[\mathbf{\Omega},\mathbf{\Pi},\boldsymbol{B},\boldsymbol{V};\tau) = \int d^3x\left(-\tfrac{i}{4}\left(a_{\Xi}^{-1}\right)^{ij}\Omega_i\Omega_j - \Xi_0^i\Omega_i\right) + \\ &+\int d^3x\left[-\tfrac{i}{4}\left(\left(a_{\Theta}^{-1}\right)^{\ell j} + \left(a_{\Delta}^{-1}\right)^{ab}\epsilon^{kj}{}_{a}\epsilon^{m\ell}{}_{b}B_kB_m\right)\Pi_\ell\Pi_j - \left(\Theta_0^i + \epsilon^{k\ell i}B_k\Delta_{0\ell}\right)\Pi_i\right]. \end{aligned} \tag{24}$$

The *“noise” Lagrangian* L_C has a density that is *local in time and space*, and this makes the theory handleable in its form.

The *phase space evolution kernel* A reads:

$$\begin{aligned} &A[\mathbf{\Omega},\mathbf{\Pi},\boldsymbol{B},\boldsymbol{V};t_0,t) = \\ &= N_0(t_0,t)\, e^{\int_{t_0}^{t}d\tau\int d^3x\left(-\frac{1}{4}\left(a_{\Xi}^{-1}\right)^{ij}\Omega_i\Omega_j + i\left(\Xi_0^i - \dot{B}^i - B^i\partial_jV^j - V^j\partial_jB^i + B^j\partial_jV^i\right)\Omega_i\right)} \\ &e^{\int_{t_0}^{t}d\tau\int d^3x\left[-\frac{1}{4}\left(\left(a_{\Theta}^{-1}\right)^{\ell j} + \left(a_{\Delta}^{-1}\right)^{ab}\epsilon^{kj}{}_{a}\epsilon^{m\ell}{}_{b}B_kB_m\right)\Pi_\ell\Pi_j + i\left(\Theta_0^i + \epsilon^{k\ell i}B_k\Delta_{0\ell} - \dot{V}^i - V^j\partial_jV^i + V^i\partial_jV^j\right)\Pi_i\right]} \end{aligned} \tag{25}$$

The *configuration space evolution kernel* $\mathcal{A}[\boldsymbol{B},\boldsymbol{V};t_0,t)$ will be calculated as

$$\mathcal{A}[\boldsymbol{B},\boldsymbol{V};t_0,t) = N_0(t_0,t)\, e^{-i\int_{t_0}^{t}d\tau L[\boldsymbol{B},\boldsymbol{V},\partial_t\boldsymbol{B},\partial_t\boldsymbol{V};\tau)}, \tag{26}$$

being L referred to as the total stochastic Lagrangian. When this is done, and once the quantities

$$\begin{cases} \Xi_0^i - \dot{B}^i - B^i \partial_j V^j - V^j \partial_j B^i + B^j \partial_j V^i = f_B^i, \\ \Theta_0^i - \epsilon^{k\ell i} B_k \Delta_{0\ell} - \dot{V}^i - V^j \partial_j V^i + V^i \partial_j V^j = f_V^i, \\ \left(a_\Theta^{-1}\right)^{\ell j} + \left(a_\Delta^{-1}\right)^{ab} \epsilon^{kj}{}_a \epsilon^{m\ell}{}_b B_k B_m = \mathcal{G}_{\Theta\Delta}^{\ell j} \end{cases} \tag{27}$$

are defined, L may be split into a field-dependent addendum

$$L_{\text{field}}\left[\boldsymbol{B}, \boldsymbol{V}, \partial_t \boldsymbol{B}, \partial_t \boldsymbol{V}; \tau\right) = \\ = -i \int d^3x \ln\left(\sqrt{\det\left\|\mathcal{G}_{\Theta\Delta}^{ij}\right\|}\right) - i \int d^3x \left[a_\Xi^{ij} f_{Bi} f_{Bj} + \left(\mathcal{G}_{\Theta\Delta}^{-1}\right)^{ij} f_{Vi} f_{Vj}\right] \tag{28}$$

and a field-independent one

$$\lambda(\tau) = i \int d^3x \ln\left(64^3 \pi^3 \sqrt{\det\left\|a_\Xi^{ij}\right\|}\right). \tag{29}$$

$\lambda(\tau)$ does not give any contribution to the ensemble average $\langle \mathcal{F} \rangle$ in Eq. (4).

As the stirring forces are split into their "classical" part plus the fluctuations

$$\boldsymbol{\Xi} = \boldsymbol{\Xi}_0 + \boldsymbol{\xi},\ \boldsymbol{\Delta} = \boldsymbol{\Delta}_0 + \boldsymbol{\delta},\ \boldsymbol{\Theta} = \boldsymbol{\Theta}_0 + \boldsymbol{\theta} \tag{30}$$

one can appreciate

$$\boldsymbol{f}_B \overset{\circ}{=} -\boldsymbol{\xi},\ \boldsymbol{f}_V \overset{\circ}{=} -\boldsymbol{\delta} \times \boldsymbol{B} - \boldsymbol{\theta} \tag{31}$$

where the symbol $\overset{\circ}{=}$ means *equal along the motion*. The quantities $\boldsymbol{f}_B$ and $\boldsymbol{f}_V$ defined in Eq. (27) are a measure of *how much the stirring forces depart from their "classical" values*. Due to the definitions of $\mathcal{G}_{\Theta\Delta}$ and to the role of a_Ξ, the scalar expressions $a_\Xi^{ij} f_{Bi} f_{Bj}$ and $\left(\mathcal{G}_{\Theta\Delta}^{-1}\right)^{ij} f_{Vi} f_{Vj}$ actually measure the ratio of the depart of the stirring forces from their "classical value" to the width of their Gaussian distribution. If good reasons exist to assume these ratios to be small, perturbative expansions may be done in the exponentiation of L_{field} giving the stochastic evolution operator.

Note finally that the stochastic Lagrangian formed by the addenda in Eqs. (28) and (29) is an imaginary quantity, that is no surprise: indeed, $\mathcal{A}$ must be real, so that $\mathcal{P}_{\boldsymbol{B}^{(i)} \oplus \boldsymbol{V}^{(i)} \to \boldsymbol{B}^{(f)} \oplus \boldsymbol{V}^{(f)}}$ is real too. This could be "fixed" simply giving another definition in Eq. (7), where the imaginary unit could be consistently omitted.

It is sensible to assume ζ to be isotropic, and *the conductivity gradient to be negligible* with respect to the current curl, as done in Priest (2001). Considering the explicit form for $\boldsymbol{f}_B$ and $\boldsymbol{f}_V$, using Ampére's Law and working under the incompressible flow hypothesis, one may write the field-dependent part of the stochastic Lagrangian as:

$$L'_{\text{field}}\left[\boldsymbol{B}, \boldsymbol{V}, \partial_t \boldsymbol{B}, \partial_t \boldsymbol{V}; \tau\right) = -i \int d^3x \ln\left(1 + \frac{a_\Theta}{a_\Delta} |\boldsymbol{B}|^2\right) + \\ -i \int d^3x a_\Xi \left|\partial_t \boldsymbol{B} + (\boldsymbol{V} \cdot \boldsymbol{\partial}) \boldsymbol{B} - (\boldsymbol{B} \cdot \boldsymbol{\partial}) \boldsymbol{V} + \frac{\zeta_0}{\mu_0} \boldsymbol{\partial} \times (\boldsymbol{\partial} \times \boldsymbol{B})\right|^2 + \\ -i \int d^3x \frac{a_\Theta}{1 + \frac{a_\Theta}{a_\Delta}|\boldsymbol{B}|^2} \left\{\left|\partial_t \boldsymbol{V} + (\boldsymbol{V} \cdot \boldsymbol{\partial}) \boldsymbol{V} + \frac{1}{\rho_0}\left(\boldsymbol{\partial} p_0 + \frac{\boldsymbol{\partial} B^2}{2\mu_0}\right) - \frac{1}{\mu_0 \rho_0} (\boldsymbol{B} \cdot \boldsymbol{\partial}) \boldsymbol{B}\right|^2 + \\ + \frac{a_\Theta}{a_\Delta}\left[\left(\partial_t \boldsymbol{V} + (\boldsymbol{V} \cdot \boldsymbol{\partial}) \boldsymbol{V} + \frac{1}{\rho_0}\left(\boldsymbol{\partial} p_0 + \frac{\boldsymbol{\partial} B^2}{2\mu_0}\right) - \frac{1}{\mu_0 \rho_0} (\boldsymbol{B} \cdot \boldsymbol{\partial}) \boldsymbol{B}\right) \cdot \boldsymbol{B}\right]^2\right\}. \tag{32}$$

The distributions ζ_0, ρ_0 and p_0 are defined as those corresponding to the expected values $\boldsymbol{\Xi}_0$, $\boldsymbol{\Delta}_0$ and $\boldsymbol{\Theta}_0$.

In order to use the expression Eq. (32) for $\mathcal{P}_{\boldsymbol{B}^{(i)} \oplus \boldsymbol{V}^{(i)} \to \boldsymbol{B}^{(f)} \oplus \boldsymbol{V}^{(f)}}(t_0, t)$ the correct functional measure must be found. In their book Feynman and Hibbs (1965) give examples in which the functional measure is determined so to render self-consistent the equation of motion of the "kernel" corresponding to $\mathcal{A}[\boldsymbol{B}, \boldsymbol{V}; t_0, t)$. In the quantum case the kernel satisfies the Schrödinger equation, while in a stochastic theory one should determine the measure so to let a Fokker-Planck equation be defined for $\mathcal{A}[\boldsymbol{B}, \boldsymbol{V}; t_0, t)$. A special discussion is needed if the stirring forces show memory effects or non-local correlations, instead: apparently, no "Schrödinger-like" equation can be given for the kernel $\mathcal{A}$ (Phythian, 1979).

5 Summary and comments

In the present paper the irregular space- and time-variability of resistivity, density and pressure in turbulent plasmas gives a motivation to construct a stochastic field theory that should be able to describe the out-of-equilibrium statistical dynamics of some resistive MHD systems. This should predict the effects of sudden, intermittent fluctuations of the medium yielding impulsive phenomena, without renouncing to representing the plasma as a continuum.

Everything is based on the knowledge of the local probability density function $\mathcal{P}_{\rm dyn}$ of ζ, ρ and p and on the construction of the stirring force functional $\mathcal{Q}$ from it. This $\mathcal{P}_{\rm dyn}$ should be determined from the dynamical theory of the plasma particles. Note the very crucial role played by the PDFs of ζ, ρ and p in all this framework: this is a way of taking into account all the possible fluctuations of the medium, included the "rare events" that might be driving the very long time behaviour of the system, and hence cannot be neglected.

Limitations of the proposed scheme can be already recognized. First of all, no discussion has been even initiated yet about the existence and convergence of all the quantities involved. There is an apparent "necessity" of making the choice Eq. (12) in order to follow the scheme traced in Phythian (1977), at least as far as the authors have been able to understand, due to the necessity of passing through the definitions of as many "stochastic momenta" as the additive stirring forces in the Langévin Equations. It could be useful to extend the reasoning presented here to other forms of the Langévin equations so to avoid the positions Eq. (12) and work directly with ζ, ρ and p as stirring forces in Eq. (9).
It is also important to mention that the problem of defining a good functional measure is still to be examined, by studying the consistency condition of a Fokker-Planck equation for $\mathcal{A}$, starting for instance with the Lagrangian Eq. (32), obtained under drastically simplifying hypotheses.

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