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Consequences of entropy bifurcation in non-Maxwellian astrophysical environments

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Abstract. Non-extensive systems, accounting for long-range interactions and correlations, are fundamentally related to non-Maxwellian distributions where a duality of equilibria appears in two families, the non-extensive thermodynamic equilibria and the kinetic equilibria. Both states emerge out of particular entropy generalization leading to a class of probability distributions, where bifurcation into two stationary states is naturally introduced by finite positive or negative values of the involved entropic index kappa. The limiting Boltzmann-Gibbs-Shannon state (BGS), neglecting any kind of interactions within the system, is subject to infinite entropic index and thus characterized by self-duality. Fundamental consequences of non-extensive entropy bifurcation, manifest in different astrophysical environments, as particular core-halo patterns of solar wind velocity distributions, the probability distributions of the differences of the fluctuations in plasma turbulence as well as the structure of density distributions in stellar gravitational equilibrium are discussed. In all cases a lower entropy core is accompanied by a higher entropy halo state as compared to the standard BGS solution. Data analysis and comparison with high resolution observations significantly support the theoretical requirement of non-extensive entropy generalization when dealing with systems subject to long-range interactions and correlations.

1 Introduction

Power-law behavior as manifestation of fractal or multifractal structures is found in a great variety of complex phenomena in different scientific fields. A novel context of description is based on entropy generalization and maximization of the corresponding particular entropy function. Such a generalization is an intrinsic nonlinear process where the resulting power-law distributions follow in a natural way. Nature is per se nonextensive and complex since any member of an ensemble of particles, e.g. the solar wind, the interstellar medium or star clusters, is subject to electromagnetic and/or gravitational interactions. With regard to the two limits, the crystal as system of maximum order described by pure geometry and, vice versa, a thermalized gas of independently moving particles described by standard Boltzmann-Gibbs-Shannon (BGS) statistics, nature appears somewhere between. In extensive systems no interactions or correlations are present and the BGS logarithmic entropy yields the Maxwellian distribution. On the other hand, the members of nonextensive systems are subject to long-range interactions and couplings, accessible by a generalized entropy functional, where the corresponding power-law distributions depend on a specific parameter, the entropic index.

Leptokurtic, long-tailed probability distribution functions (PDF's) subject to a non-Gaussian core and a pronounced halo are a persistent feature in a variety of different astrophysical environments. Those include the thermo-statistical properties of the interplanetary medium where the electron, proton and even heavy ion velocity space distributions show ubiquitously suprathermal halo patterns (see Mendis and Rosenberg (1994) for a general review, or Leubner (2000); Leubner and Schupfer (2001) and references therein), well described by the empirical family of κ-distributions, a power law in particle speed and first recognized by Vasyliunas (1968). In continuation, significant progress was provided by Treumann (1999a,b) who developed a kinetic theory, demonstrating that power-law velocity distributions are a particular thermodynamic equilibrium state. The empirical family of κ-distribution functions was linked to power-law distributions derived in the context of nonextensive statistics by Leubner (2000, 2004a,b), thus providing the hitherto missing theoretical foundation of non-thermal interplanetary velocity distributions.

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Moreover, also the PDF's of the turbulent fluctuations of the magnetic field strength, density and velocity field differences in space and astrophysical plasmas (Leubner and Vörös, 2005a,b) show pronounced leptokurtic cores and extended tails on small scales. Since high resolution in situ satellite observations are available detailed analyses of the PDFs of the solar wind plasma are of considerable interest to study intermittency and multi-scale statistical properties in fully developed turbulence. The characteristics of the spectral properties of fluctuations in the incompressible interplanetary medium were provided in classical statistical theory via the phase space distribution obtained from ideal MHD invariants by Matthaeus and Goldstein (1982) followed by a confirmation of the existence of solar wind multi-fractal structures (Burlaga, 1991, 1992). The PDFs of the magnetic field and plasma fluctuations were analyzed and linked to intermittency of the solar wind MHD fluctuations (Marsch and Tu, 1994, 1997) along with detailed investigations of the non-Gaussian characteristics (Bruno et al., 2001; Hnat et al., 2003). WIND, ACE and Voyager observations of interplanetary multi-scale statistical properties verify that the leptokurtic, long-tailed shapes of the PDFs at small scales represent correctly the characteristics of intermittent turbulence and approach a Gaussian, reflecting a decoupled state, on large scales (Sorriso-Valvo et al., 1999; Leubner and Vörös, 2005a,b; Leubner et al., 2006), thus confirming that the probability of rare events is raised on small scales.

Remarkably, we have to add to this diversity scale invariant power-law distributions relying on self organized criticality (SOC) (Bak et al., 1988; Bak, 1996; Chapman et al., 1998; Watkins et al., 2001; Chapman and Watkins, 2001) as well as gravitationally bound astrophysical stellar systems (Nakamichi et al., 2002; Chavanis and Bouchet, 2005). Furthermore, Leubner (2005) developed in this conjunction recently a nonextensive theory representing accurately the hot plasma and dark matter (DM) density profiles in galaxies and clusters in the context of scale invariant power-law distributions. The significance and accuracy of nonextensive statistics modeling density distributions of astrophysical bound structures was confirmed by N-body and hydrodynamic simulations as well as from observations (Kronberger et al., 2006).

In all physically different situations nonextensive statistics, accounting for long-range interactions and correlations, provides a highly successful context of description whereas standard BGS statistics does not apply. In the following we summarize the concept of entropy generalization and discuss the corresponding entropy bifurcation along with the resulting power-law distributions. Nonextensive statistics is then tested on three fundamentally different physical situations: (1) the free solar wind distributions in velocity space, (2) the probability distributions of fluctuations in turbulence and (3) the radial density distributions of gravitationally clustered structures.

2 Nonextensive entropy generalization and velocity distribution functions

The classical BGS extensive thermo-statistics applies when microscopic interactions are short ranged and the environment is a continuous and differentiable manifold. Astrophysical systems, however, are generally subject to long-range interactions in a non-Euclidean, for instance fractal or multifractal environment. A suitable nonextensive generalization of the BGS entropy for statistical equilibrium was first recognized by Renyi (1955) and subsequently proposed by Tsallis (1988), suitably extending the standard additivity of the entropies to the nonlinear, nonextensive case where one particular parameter, the entropic index, characterizes the degree of nonextensivity of the system considered. A variety of subsequent analyses were devoted to clarify the mathematical and physical consequences of pseudo-additivity (Plastino et al., 1994; Tsallis, 1995; Silva et al., 1998; Almeida, 2001) where a deterministic connection between the generalized entropy and the resulting power-law functionals (Andrade et al., 2002), as well as the duality of nonextensive statistics were recognized (Karlin et al., 2002). Derived within the context of nonextensive statistics, power-law distributions provided also the missing justification for the use of the hitherto empirical, but ubiquitously observed, κ-distribution family favored in space plasma modeling from fundamental physics (Leubner, 2000; Leubner and Schupfer, 2001; Leubner, 2004a,b). The corresponding entropic index κ, as measure of the degree of long-range interactions or correlations, is not restricted to positive values and thus manifests the duality of nonextensive statistics.

Assuming that particles move independently from each other, i.e. there are no correlations present in the system considered, the BGS statistics is based on the extensive entropy measure $S_B = -k_B \sum p_i \ln p_i$, where p_i is the probability of the ith microstate, k_B is Boltzmann's constant and S_B is extremized for equiprobability. This entropy implies isotropy of the velocity directions and thus appears as additive quantity yielding the standard Maxwellian distribution function. Accounting for long-range interactions requires to introduce correlation within the system, which is done fundamentally in the context of nonextensive entropy generalization leading to scale-free power-law PDFs. Considering, as example, two sub-systems A and B one can illuminate nonextensivity by the property of pseudo-additivity of the entropies such that

$$S_\kappa(A+B) = S_\kappa(A) + S_\kappa(B) + \frac{k_B}{\kappa} S_\kappa(A) S_\kappa(B) \tag{1}$$

where the entropic index κ quantifies the degree of nonextensivity in the system. For $\kappa=\infty$ the last term on the right hand side cancels leaving the additive terms of the standard BGS statistics. Hence, nonlocality is introduced by the nonlinear term accounting for correlations between the subsystems. In general, the pseudo-additive, κ-weighted term may assume positive or negative definite values indicating a nonextensive entropy bifurcation. Obviously, nonextensive systems are subject to a dual nature since positive κ-values imply the tendency to less organized states where the entropy increases, whereas negative κ-values provide states of a higher level of organization and decreased entropy, as compared to the BGS state, see e.g. Leubner (2004a, 2005).

The general nonextensive entropy is consistent with the example in Eq. (1) and reads (Tsallis, 1988; Leubner, 2004a)

$$S_\kappa = \kappa k_B (\sum p_i^{1-1/\kappa} - 1) \tag{2}$$

where $\kappa = \infty$ represents the extensive limit of statistical independence. Consequently, the interaction term in Eq. (1) cancels recovering with respect to Eq. (2) the classical BGS entropy. A further generalization of Eq. (2) for complex systems was provided by Milovanov and Zelenyi (2000), where appropriate higher order terms in the entropy appear. Once the entropy is known the corresponding probability distributions are available.

In Maxwells derivation the velocity components of the distribution $f(v)$ are uncorrelated so that lnf can be expressed as a sum of the logarithms of the one dimensional distribution functions. In nonextensive systems one needs to introduce correlations between the components. Extremizing the entropy Eq. (2) under conservation of mass and energy the resulting distribution function in velocity space reads

$$f^{\pm} = A^{\pm} \left[1 + \frac{1}{\kappa} \frac{v^2}{v_t^2} \right]^{-\kappa} \tag{3}$$

where v_t corresponds to the mean energy or thermal speed. Hence, the exponential probability function of the Maxwellian gas of an uncorrelated ensemble of particles is replaced by the characteristics of a scale invariant power-law where the sign of κ, indicated by superscripts, governs the corresponding entropy bifurcation. We note that the distribution Eq. (3) can be derived by means of Lagrangian multiplyers without introducing a specific form for long-range interactions. Incorporating the sign of κ into Eq. (3) and performing the normalization separately for positive and negative κ-values generates a dual solution subject also to two different, κ-dependent normalizing factors $A^{\pm}(\kappa)$.

The entropy bifurcation appears also in higher order moments yielding for instance κ-dependent generalized temperatures. Furthermore, the positive solution is restricted to $\kappa > 3/2$ whereas the negative solutions are subject to a cut off in the distribution at $v_{\max} = v_t \sqrt{\kappa}$, for details see (Leubner, 2004a). Both functions, f^+ and f^- in Eq. (3) approach one and the same Maxwellian as $\kappa \rightarrow \infty$. Figure 1 demonstrates schematically the non-Maxwellian behavior of both, the suprathermal halo component and the less pronounced core distribution, subject to finite support in velocity space, where the case $\kappa = \infty$ recovers the Maxwellian equilibrium distribution.

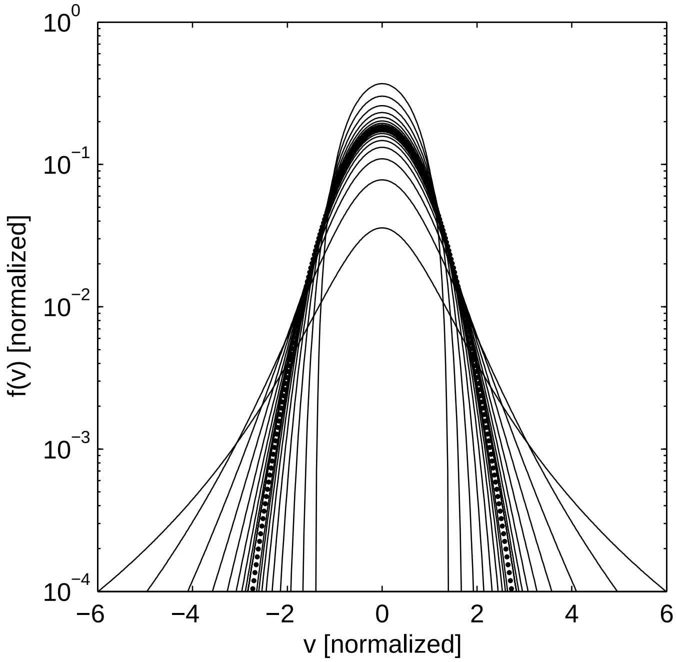

Fig. 1. A schematic plot of the characteristics of the nonextensive bi-kappa distribution family: with κ=3 the outermost and innermost curve correspond to the halo f^+ and core f^- distribution fraction. For increasing κ-values both sets of curves merge at the same Maxwellian limit, indicated as dotted line, f^+ from outside and f^- from inside. $f(v)$ and v are normalized to the maximum of the Maxwellian and to the thermal speed, respectively.

Next, we construct a unique nonextensive distribution subject to the constraints: (a) the distribution approaches one and the same Maxwellian as $\kappa \rightarrow \infty$, (b) a unique, global distribution must be definable by one single density and a unique temperature and (c) upon variation of the coupling parameter κ particle conservation and adiabatic evolution are required, such that a redistribution in a box (a source free environment) can be performed. Subject to these constraints the appropriate mathematical functional, representing observed core-halo (ch) structures in nonextensive astrophysical environments, is available from the elementary combination $f_{\mathrm{ch}} = B_{\mathrm{ch}}(f^+ + f^-)$, B_{ch} being a proper normalization constant. In this context the velocity space “bi-kappa distribution”, compatible with nonextensive entropy generalization and obeying the above constraints, reads

$$f_{\mathrm{ch}} = \frac{N}{\pi^{1/2} v_t} G(\kappa) \left\{ \left[1 + \frac{1}{\kappa} \frac{v^2}{v_t^2} \right]^{-\kappa} + \left[1 - \frac{1}{\kappa} \frac{v^2}{v_t^2} \right]^{\kappa} \right\} \tag{4}$$

The last term on the right-hand side denotes an expression subject to a thermal cut off at the maximum allowed velocity $v_{\max} = \sqrt{\kappa} v_t$, which limits also required integrations, see Fig. 1. For details regarding the corresponding second

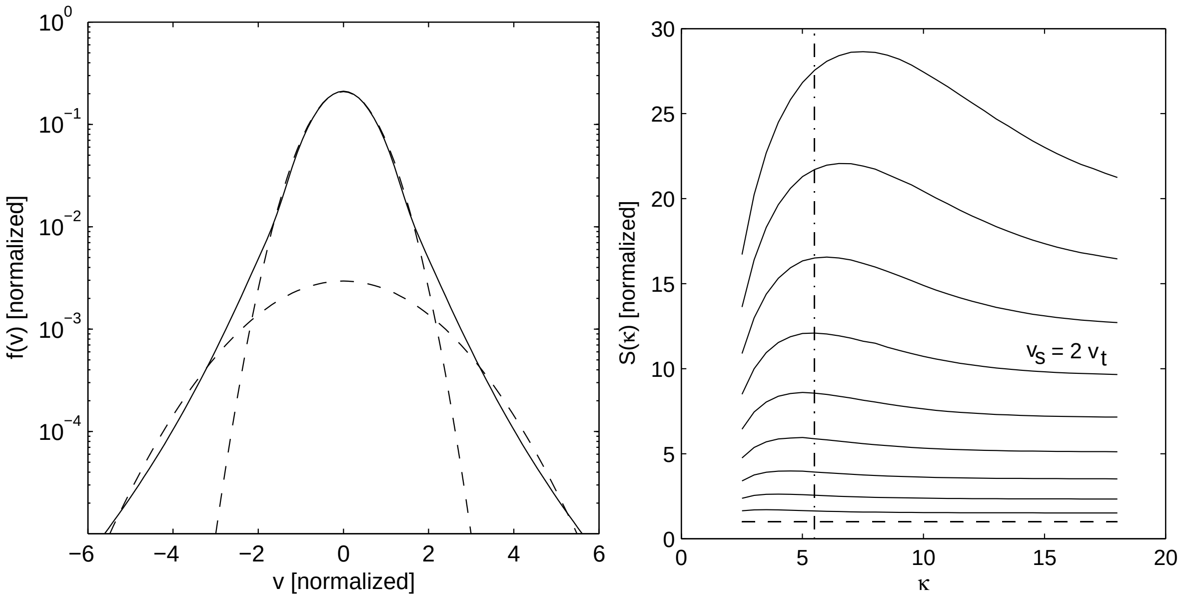

Fig. 2. Left: Reconstruction of a double-Maxwellian fit to interplanetary core-halo electron structures (dashed lines) as compared to the single, one parameter nonextensive distribution (solid line). Normalization as in Fig1. Right: The $\kappa-$dependent entropy function for increasing peak separation scale from bottom to top. The sequence of curves belong to stepwise increasing v_s in thirds of the thermal velocity and the dashed line indicates the Maxwellian reference with zero peak separation. The dashed-dotted line shows the entropy maximum for $v_s \sim 2v_t$ and relates therefore the mean observed peak separation at $v_s = 1.17v_A = 1.97v_t$ to a value of $\kappa = 5.5$. $S(\kappa)$ is normalized to one, the Maxwellian reference.

moments or generalized temperatures see Leubner (2004a,b). The function $G(\kappa)$ is defined by

$$G(\kappa) = \left[\frac{\kappa^{1/2}\Gamma(\kappa - 1/2)}{\Gamma(\kappa)} + \frac{\kappa^{1/2}\Gamma(\kappa + 1)}{\Gamma(\kappa + 3/2)} \right]^{-1} \tag{5}$$

from the normalization of f_{ch} and is subject to a particular weak κ-dependence where $G(\kappa) \sim 1/2$, see Leubner (2004a) for a graphical illustration and discussion. Hence, the normalization is independent of the parameter κ and the factor $1/2$ reflects consistently the superposition of the two counter-organizing contributions in Eq. (4). For $\kappa = \infty$, $G(\kappa) = 1/2$ and the power laws in the brackets of the right hand side of Eq. (4) turn each into the same Maxwellian exponential and the resulting factor 2 cancels with $G(\kappa)$.

The duality of equilibria in nonextensive statistics is manifest in two families, the nonextensive thermodynamic equilibria and the kinetic equilibria, where both families are related via the nonextensive parameter by $\kappa' = -\kappa$ (Karlin et al., 2002; Leubner, 2005). κ' and κ denote the corresponding entropic index of the particular family where the transformation $q = 1 - 1/\kappa$ for the transition between the Tsallis q-notation and the κ-formalism used here is applied (Leubner, 2004a). Positive κ-values are related to the stationary states of thermodynamics and negative κ-values to kinetic stationary states. The limiting BGS state for $\kappa = \infty$ is therefore characterized by self-duality. The nonextensive parameter κ finds also a physical interpretation in terms of the heat capacity of a medium (Almeida, 2001). A system with $\kappa > 0$ represents an environment with finite positive heat capacity and vice versa, for $\kappa < 0$ the heat capacity is negative. Negative heat capacity is a typical property of self-interacting systems, see e.g. Firmani et al. (2000). Moreover, contrary to thermodynamic systems where the tendency to dis-organization is accompanied by increasing entropy, self-interaction tends to result in structures of a higher level of organization and decreased entropy. Consistently, "core" refers to negative definite κ and "halo" to positive definite κ-values and the corresponding distribution families merge for $\kappa \to \infty$ into the extensive, selfdual state.

2.1 Interplanetary velocity distribution functions

Clear signature of persistent core-halo solar wind electron structures were provided by Ulysses and WIND observations (Maksimovic et al., 1997a; Pierrard et al., 1999). Based on Ulysses detections Maksimovic et al. (2000) studied solar wind core-halo electron density and temperature profiles, performing also a classical two-Maxwellian fit to observed distributions. The Maxwellian core fit resembles accurately the observed characteristics but, due to the concave distribution slope of the Maxwellian, only a rough representation of the measured convex high-energy tail structure (see Fig. 1 of their study) is provided. Figure 2, left panel, shows a reconstruction of the two-Maxwellian fit as compared to the unique nonextensive bi-kappa fit, demonstrating clearly the advantage of the nonextensive representation, in particular with regard to the strongly pronounced core continuing smoothly into the observed convex halo distribution shape. Further significant support for the nonextensive bi-kappa approach is provided from solar wind electron observations (Maksimovic et al., 1997b; Pierrard et al., 1999) where matching shortcomings of Maxwellian fits and empirical positive-kappa distribution fits are corrected with the bi-kappa core-halo approach, in particular due to the enhanced distribution maximum and a reduced distribution width of the nonextensive core for particle energies just below the core-halo transition, for details see Leubner (2004a,b).

As fundamental advantage, best fits are obtained on the basis of a unique core/halo density and temperature according to Eq. (4) providing one single distribution, which represents accurately the observed electron structures, where the entropic index κ remains as the only distribution shaping parameter. Theoretically, any separation of an observed velocity distribution, for instance into two different Maxwellians and therefore subject to different temperatures and densities in the core and halo, respectively, is questionable since density and temperature are defined as moments of the entire distribution.

In contrast to interplanetary electron distributions proton velocity space structures typically exhibit a clear core-halo peak separation v_s along the ambient magnetic field where the separation scale was suggested to average around 1.4 times the local Alfvén speed v_A (Marsch et al., 1982). In view of the proposed correlation between peak separation and the Alfvén speed, or relative core-halo drift, it must be emphasized that the data are subject to a significant spread of $(0.58...1.73)v_A$ around the favored value of $1.4v_A$. Therefore much effort was spent over two decades to clarify the origin of the persistent solar wind core-halo separation, observed in particular in high speed streams.

We refer to HELIOS observations of proton velocity distributions between 0.3 and 1 AU (Marsch et al., 1982), later theoretically supported by Leubner and Viñas (1986) providing in view of cyclotron instability analysis a series of accurate analytical two-dimensional representations of the detected double humped structures. This context assists to analyze the dependence of the nonextensive entropy function on the entropic index κ and the relative drift speed where the entropy $S{=}{-}k_B \int f(\mathbf{v},\mathbf{r}) \ln[f(\mathbf{v},\mathbf{r})]d\mathbf{v}d\mathbf{r}$ is evaluated following Collier (1995). Using the Boltzmann entropy for the evaluation of non-thermal distributions introduces an approximation, a subject presently under investigation. Starting with zero model peak separation and increasing stepwise the relative core-halo drift as parameter, see Fig. 2, right panel, associates through the the entropy maximum the mean peak separation scale to a particular value of κ. Most significantly, the entropy maximum relates a peak separation of $v_s{\sim}2v_t$ to $\kappa{\sim}5$, a value consistent with solar wind observations. For a given κ-value it is therefore possible to deduce via the maximum entropy condition the corresponding relative core-halo drift speed of interplanetary velocity distributions. This enables one to construct typically observed double-humped distributions directly from the knowledge of the particle density and plasma temperature, since the maximum entropy condition relates the relative drift v_s to the corresponding nonextensive index κ. κ-distributions subject to low κ-values (κ=3...6) represent best the "normal" situation in space plasmas.

2.2 Probability distributions in turbulence

The analysis of probability distribution functions (PDFs) is of considerable interest to study intermittency and multi-scale statistical properties in fully developed turbulence in the solar wind plasma where high resolution in situ observations are available. We test in the following the relevance of the nonextensive, global bi-kappa PDF Eq. (4), adapted to study the observed scale dependence of the PDFs of the differences of magnetic field fluctuations in the intermittent, turbulent interplanetary medium.

For this purpose we relate the energy levels E_i of the turbulent spectrum to the corresponding kinetic energy of velocity differences $\delta v(t){=}v(t{+}\tau){-}v(t)$ between two points of separation τ, allowing to transform the 1-D Maxwellian particle distribution of mean energy v_t into the mathematical form of a Gaussian of variance σ. Upon normalizing the one-dimensional bi-kappa particle distribution Eq. (4) to unity and assigning the distribution variance σ to the thermal spread v_t the "Maxwellian form" of the bi-kappa distribution transforms into a "Gaussian form" of a global bi-kappa PDF for statistical analyses in turbulence as

$$P_{ch}(\kappa,\sigma) = \frac{1}{2\sqrt{\pi}\sigma}\left\{\left[1+\frac{\delta X^2}{\kappa\sigma^2}\right]^{-\kappa} + \left[1-\frac{\delta X^2}{\kappa\sigma^2}\right]^{\kappa}\right\} \quad (6)$$

The spatial separation scale is characterized in common notation by the differences $\delta X(t){=}X(t{+}\tau){-}X(t)$, $X(t)$ denoting any characteristic solar wind variable at time t and τ is the time lag. As previously, κ characterizes physically the degree of nonextensivity or nonlocality in the system, thus being a measure of the degree of organization or intermittency, respectively, (Leubner and Vörös, 2005a) and σ denotes the distribution variance. In analogy, Fig. 1 illuminates now that large values of δX correspond to the tails of the distribution, represented by the first term on the right-hand side of Eq. (6), whereas small differences are related to the distribution core and are modeled primarily by the second term of Eq. (6). As $\kappa\to\infty$ the bi-kappa distribution $P_{\mathrm{ch}}(\kappa,\sigma)$ approaches a single Gaussian.

The basic assumption for deriving the velocity space bi-kappa distribution was the pseudo-additivity of the entropies of particle sub-systems expressed by Eqs. (1) and (2). It is important to recognize that the same type of expression for a bi-kappa distribution is obtained, if we assume instead of interacting particles interacting coherent structures with

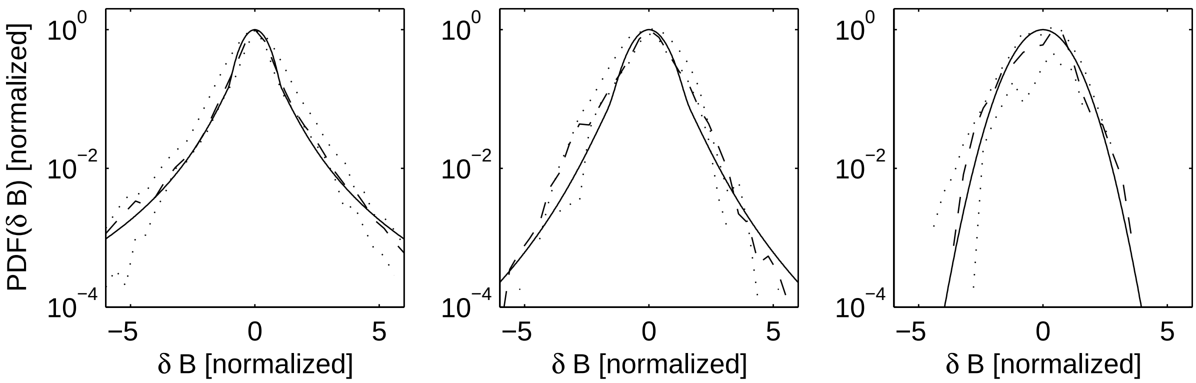

Fig. 3. The PDF of the increments of observed ACE magnetic field fluctuations for $\tau=100$ and a resolution of 16s as compared to the bi-kappa function with $\kappa=1.8$. Based on the same data the central panel provides the characteristics for increased $\tau=2000$ where κ assumes a value of 3.0 for the best representation. The PDF of large-scale magnetic field fluctuations, $\tau=10000$, are well modeled by a Gaussian with $\kappa=\infty$, right panel. The dotted lines correspond to the standard deviations of observational data, the PDFs are normalized to the maximum value, δB to the standard deviation and τ is normalized to the time resolution.

the same pseudo-additive property. In the context of MHD, non-propagating multi-scale coherent structures or flux tubes can interact, deform and produce new sites of nonpropagating fluctuations. Coherent structures of the same polarity merge into a structure with lower local energetic states, while structures of opposite polarities may repel each other (Chang, 1999; Bruno and Carbone, 2005). These coherent structures can be considered as discrete interacting “particles” in MHD flows, validating the analogy to the kinetic level of PDFs (Leubner and Vörös, 2005b). Moreover, passive scalars as the magnetic field B, discussed here, follow the dynamics of v or δv (Vörös et al., 2006).

Multi-scale redistribution of energy – a basic feature of turbulent flows - appears, where interacting coherent structures may also reduce the entropy of the system, leading to negative κ-values. At the same time, turbulence enhances dissipation and mixing of the plasma, which increases entropy and is described in terms of positive κ-values. Both processes are incorporated in Eq. (6).

Based on the nonextensive two parameter bi-kappa distribution Eq. (6) we compare the PDFs of magnetic field fluctuations, obtained from slow and fast solar wind data, with particular attention to the scale dependent changes of the two physically interpretable parameters (κ, σ) involved. The presence of discontinuities or shocks is a problem investigated elsewhere (Vörös et al., 2006) and is not considered here. For each data set the magnetic field increments were calculated at a given time lag τ by $\delta B(t)=B(t+\tau)-B(t)$, normalized to the standard deviation and followed by the computation of the probability distribution function (histogram). $\delta B(t)$ represents the characteristic fluctuations at a particular time scale τ or, equivalently, across eddies of size $l=v\tau$, v being the solar wind speed. The dimensionless τ is multiplied by the time resolution to generate an effective time lag. Changing τ allows then to analyze the statistical features of fluctuations on different scales. For this analysis magnetic field data, available with 16 s from the ACE magnetic field experiment are used (Smith et al., 1998).

Fig. 3 demonstrates that the scale dependence of the PDFs in the low speed solar wind are accurately represented via the tuning parameters κ and σ of the bi-kappa functional Eq. (6). The strong non-Gaussianity of the PDFs of small scale fluctuations must be associated physically with long-range interactions and correlations due to the underlying nonextensive context. Undisturbed solar wind ACE magnetic field amplitude data of 16 s time resolution are analyzed where the evolution of magnetic field fluctuations is subject to a two point separation scale of $\tau=100, 2000$ and 10000. The corresponding best fits of the bi-kappa distribution are obtained for $\kappa=1.8, 3$ and ∞ and the dotted lines refer to the standard deviation of observational data. The accuracy of the bi-kappa distribution fits clearly indicates that non-locality in turbulence, when introduced theoretically by long-range interactions through a nonextensive entropy generalization, provides a precise representation of the observed PDFs characterizing the intermittency of the magnetic field fluctuations at all scales.

The three panels in Fig. 4 show PDFs of high speed associated magnetic field magnitude fluctuations where the two-point statistics is performed for the scales $\tau=40, 400$ and 10000. Note that in comparison to low speed data the degree of nonextensivity does not change during high speed intervals, leaving κ almost constant over the range of scales ($\tau=40$ to 400), where only for decoupled Gaussian state ($\tau=10000$) κ approaches ∞. In high speed environments the mean energy represented by σ needs to be adjusted for accurate fits. This indicates physically that the abundance of large scale energy content in high speed flows facilitates to maintain the degree of nonextensivity and self-organisation unchanged over the considered range of scales, for details see Leubner et al. (2006).

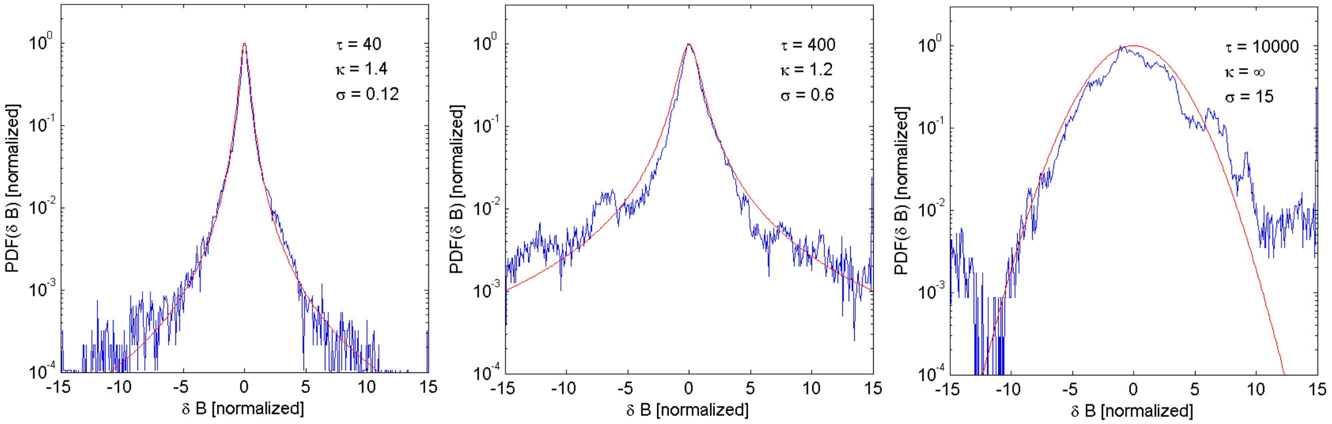

Fig. 4. The PDF of the increments of observed ACE high-speed associated magnetic field magnitude fluctuations (16 s time resolution) normalized to the maximum value. Left panel: fluctuations at the scale τ=40 as compared to the bi-kappa function with κ=1.4 and σ=0.12; Medium panel: τ=400, κ=1.2 and σ=0.6; Right panel: Gaussian fit with τ=10000, $\kappa=\infty$ and σ=15. τ is normalized to the time resolution and δB to the standard deviation.

As ordering parameter κ accounts for nonlocality or correlations within the system. Highly correlated turbulent conditions characterized by κ-distributions represent stationary states far from equilibrium where a generalization of the BGS entropy, as measure of the level of organization or intermittency, applies (Goldstein and Lebowitz, 2004; Treumann et al., 2004). Physically this can be understood considering a system at a certain nonlinear stage where turbulence may reach a state of high energy level, balanced by turbulent dissipation. In this environment equilibrium statistics can be extended to dissipative systems, approaching a stationary state beyond thermal equilibrium (Gotoh and Kraichnan, 2002).

2.3 Density distributions in clustered structures

To date only a few attempts provide physically motivated models for density profiles of astrophysical clusters. Early analytical analysis (Gunn and Gott, 1972) for the collapse of density perturbations was subsequently further studied (Hoffman, 1988) and based on infall models (Williams et al., 2004; Ascebar et al., 2004).

In practice, dark matter (DM) and hot plasma density profiles, as observed in galaxies and clusters or generated in simulations, are widely modeled by empirical fitting functions. The phenomenological $\beta-$model (Cavaliere and Fusco-Femiano, 1976), provides a reasonable representation of the hot plasma density distribution of clustered structures, further improved by the double β-model with the aim of resolving the $\beta-$discrepancy (Bahcall and Lubin, 1994). Similarly, the radial density profiles of DM halos are analyzed primarily with the aid of phenomenological fitting functions, thus lacking physical support as well (Burkert, 1995; Navarro et al., 1996; Moore et al., 1998).

Since any astrophysical system is subject to long-range gravitational and/or electromagnetic interactions, this situation motivates again to introduce nonextensive statistics as physical background for the analysis of DM and hot plasma density profiles. In this context the entropy of the standard (BGS) statistics is generalized, as outlined in section 1, by the pseudo-additive κ-weighted term to mimic the degree of long-range gravitational interactions and correlations within the system.

Extremizing the generalized entropy with regard to conservation of mass and energy in a gravitational potential Ψ yields the energy distribution

$$f^{\pm}(v) = C^{\pm}\left[1 + (v^2/2 - \Psi)/(\kappa\sigma^2)\right]^{-\kappa} \tag{7}$$

As previously, the superscripts refer to the positive or negative intervals of the entropic index κ, accounting for less (+) and higher (–) organized states and thus reflecting the accompanying entropy increase or decrease, respectively (Leubner, 2005). σ represents the mean energy of the distribution and $C^{\pm}$ are the corresponding normalization constants. The density evolution of a system subject to long range interactions in a gravitational potential

$$\rho^{\pm} = \rho_0\left[1 - \Psi/(\kappa\sigma^2)\right]^{(3/2-\kappa)} \tag{8}$$

is found after integration over all velocities. Combining with Poisson's equation $\Delta\Psi = -4\pi G\rho^{\pm}$ provides a second order nonlinear differential equation, determining the radial density profiles of both components, plasma and DM in clustered structures as (Leubner, 2005)

$$\frac{d^2\rho}{dr^2}+\frac{2}{r}\frac{d\rho}{dr}-(1-\frac{1}{n})\frac{1}{\rho}(\frac{d\rho}{dr})^2-\frac{4\pi Gn}{(\frac{3}{2}-n)}\frac{\rho^2}{\sigma^2}(\frac{\rho}{\rho_0})^{-\frac{1}{n}}=0 \tag{9}$$

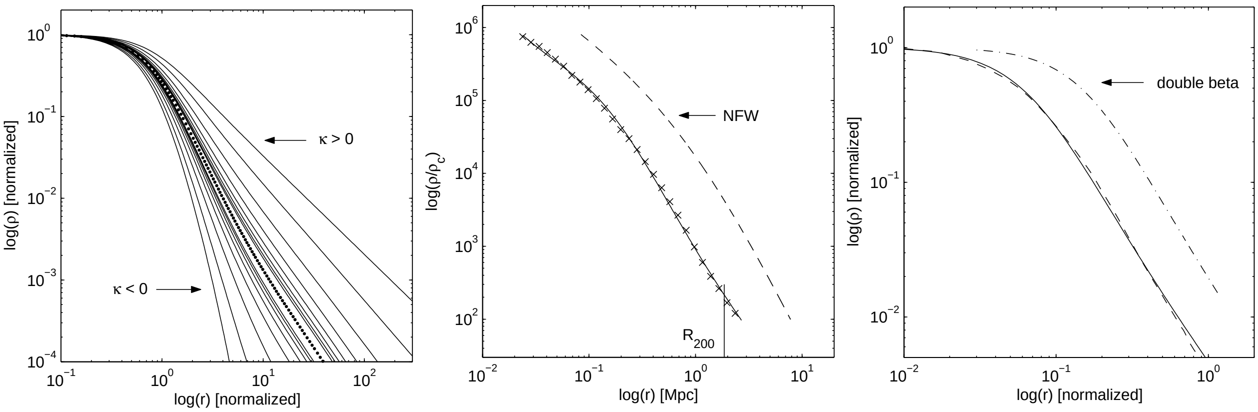

Fig. 5. Left: Nonextensive family of density profiles. The set of curves left of the bold dotted line, denoting the standard BGS case, corresponds to the DM (ρ^-) solutions and the right branch of curves to the plasma (ρ^+) distributions. For increasing $\kappa-$values both sets of curves converge to the BGS isothermal sphere solution (κ=∞, bold dotted line). The density is normalized to one, generating the corresponding radius normalization. Center: Radial DM density profile obtained from N-body simulations (crosses). The solid line shows a fit of the nonextensive theory to the data with best fitting values of κ=-15 and σ=0.12. For comparison, also the best fitting Navarro et al. (1996) profile is provided (dashed line) and shifted to the right for better visibility. R_{200} indicates the virial radius. Right: Radial plasma density profile obtained from the hydrodynamic simulations (dashed line). The solid line shows a fit of the nonextensive theory to the data with best fitting values of κ=6.5 and σ=0.086. For comparison, also the best fitting double beta model is provided (dashed-dotted line), shifted to the right for better visibility. Normalization as in the left panel of Fig. 5.

where n=3/2$-\kappa$ is introduced and corresponds to the polytropic index of stellar dynamical systems (Binney and Tremaine, 1994). As natural consequence of nonextensive entropy generalization the standard isothermal sphere profile (Binney and Tremaine, 1994) bifurcates into two distribution families controlled by the sign and value of the correlation parameter κ.

Physically, we regard the DM halo as an ensemble of self-gravitating, weakly interacting particles in dynamical equilibrium (Firmani et al., 2000; Spergel and Steinhardt, 2000) and the hot gas component as an electromagnetically interacting high temperature plasma in thermodynamic equilibrium. Hence, astrophysical clusters experience long-range gravitational and/or electromagnetic interactions leading to correlations, such that the standard BGS statistics does not apply again. As discussed previously, the duality of equilibria in nonextensive statistics appears in the nonextensive stationary states of thermodynamics subject to finite positive heat capacity and in the kinetic stationary states with negative heat capacity, a typical property of self-gravitating systems (Firmani et al., 2000), where both are related only via the sign of the coupling parameter κ. Consequently we have to assign negative κ-values, describing the lower entropy state due to gravitational interaction, to the DM component and the second branch of positive κ-values and higher entropy, as compared to the BGS self-dual state, to the hot plasma component.

The left panel in Fig. 5 illuminates schematically the radial density profile characteristics for some values of κ for both, DM below and the plasma distributions above the standard exponential BGS solution. Increasing κ values correspond to a decoupling within the system and both branches merge simultaneously in the isothermal sphere profile for κ=∞, representing the extensive limit of statistical independence in analogy to the Maxwellian limit for systems where gravitational interaction is neglected. In Fig. 5, central panel, the result of N-body DM simulations are compared with the nonextensive theoretical approach for κ=-15 and σ=0.12 indicating perfect agreement. For comparison also the best fitting Navarro et al. (1996) (NFW) profile is provided. Again, reproducing simulations precisely, the radial nonextensive plasma density distribution (solid line) is compared with the results of hydrodynamic simulations in the right panel of Fig. 5. Shifted to the right also the empirical best fitting double-beta model is shown, confirming that the nonextensive theory provides naturally a context able to solve the $\beta-$discrepancy (Bahcall and Lubin, 1994).

3 Summary and conclusions

The nonextensive entropy generalization for systems subject to long-range interactions and correlations provides a natural way to redistribute a velocity space structure such that core-halo distributions, known from a variety of astrophysical observations, are generated. Tuning only the entropic index κ a thermalized Maxwellian can be transformed into highly non-thermal features, as persistently detected for instance in the interplanetary medium. The nonextensive bi-kappa distribution in Eq. (4) offers access to understand the significantly pronounced, but normally symmetrically appearing, electron core-halo distributions. Interplanetary proton or ion structures, typically subject to a core-halo separation in velocity space, are equivalently well represented within a generalized entropy concept where the separation scale obeys a maximum entropy condition.

Furthermore, this generalization yields a two-parameter global bi-kappa function providing theoretical access to the scale dependence of the differences of fluctuations of any physical variable via the PDFs observed in astrophysical plasma turbulence. The redistribution of a Gaussian on large scales into highly non-Gaussian leptokurtic and long-tailed structures, manifest on small scales, is theoratically well described by the family of nonextensive distributions. Pseudo-additive entropy generalization provides the required physical interpretation of the parameter κ in terms of the degree of nonextensivity of the system as a measure of nonlocality or couplings, whereas the variance σ measures the mean energy in the system. The scale dependence in the slow speed solar wind is sensitive to variations of κ and in high speed streams to variations of σ, see also Leubner et al. (2006).

Finally, generalizing to self-gravitating systems the dual nature of the nonextensive theory provides also a solution to the problem of DM and plasma density distributions of clustered matter from fundamental physics, where both parameters admit again physical interpretation.

Concluding, based on a fundamental entropy principle nonextensive statistics provides naturally the power to create highly non-Maxwellian core-halo velocity distributions as observed in space plasma environments. In additon, we argue that multi-scale coupling and intermittency of the turbulent solar wind fluctuations must be related to the nonextensive character of the interplanetary medium, accounting for long-range interaction via the entropy generalization. Extending the theory to gravitationally bound structures implies also the requirement to favor the physical family of nonextensive distributions over empirical models, when fitting observed or simulated density profiles of astrophysical clusters.

Hitherto, in all three cases phenomenological models served for theoretical and data analyses and should be replaced by distributions arising from the physical approach of nonextensive statistics and the underlying fundamental entropy bifurcation.

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