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Nonlinear interaction between long inertio-gravity and rossby waves

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Abstract. The equations describing the interaction of long inertio-gravity (IG) waves with the Rossby waves are derived. Due to remarkable cancellations, the interaction is shown to be anomalously weak. As a result, an inverse cascade of turbulence produces wave condensate of large amplitude so that wave breaking with front creation can occur.

1 Introduction

Numerous observations (Warren, 1966; Vincent, 1985; Vinnichenko, 1970; Van Delden, 1992) show that frequency spectra of large-scale atmospheric and oceanic turbulence have a strong peak at the frequency of the uniform inertial oscillation (that of Foucault pendulum) $f = 2\Omega \sin\phi$. Here Ω is the rotation frequency of the planet, ϕ is the local latitude. Theory (Falkovich, 1992) explains the creation of this peak as a result of inverse turbulent cascade due to inertio-gravity waves. Coriolis parameter f is the lowest frequency of IG waves. The inverse cascade due to inertio-gravity waves has the energy spectrum $E(k) \propto k^{-7/3}$ (Falkovich and Medvedev, 1992) and it could take place for the scales larger than those of the inverse 2d energy cascade $E(k) \propto k^{-5/3}$ (Kraichnan, 1967). Since the exponent of the wave cascade (7/3) is larger than the exponent of the vortex cascade (5/3) then for sufficiently small k the wave cascade may overcome the vortex one. Such a picture is in a good agreement with the data of the atmospheric observations (Lilly and Petersen, 1983; Nastrom and Gage, 1983) that show the 5/3 spectrum until wavelengts hundreds of kilometers and for larger scales the spectrum is steeper with the exponent $2.2 \div 2.4$ that is quite close to the theoretical value of 7/3 for the wave cascade. That inverse cascade produces the condensate of uniform inertial oscillations and it is the challenge for theory to explain a feedback mechanism that stops the growth of the condensate.

It has been shown by Falkovich (1992) that nonlinear self-interaction between IG waves cannot stop the cascade or provide for an instability of condensate. The interaction of short IG waves with long Rossby waves in the equatorial region has been considered by Boyd (1973). Such an interaction also cannot stop the inverse cascade and creation of longer and longer IG waves (Boyd, 1973; Falkovich, 1992). In our consideration, we allow wavelengthes of IG and Rossby waves to be comparable and be larger than Rossby radius (dispersion scale). We treat atmosphere and ocean as a rotating shallow water. In Sect.2 we derive a truncated system of equations from the general shallow-water equations and show that IG-Rossby interaction is anomalously weak and cannot provide an instability of the condensate too. This explains the existence of large-scale anticyclonic currents in the ocean that are necessary to provide for the observable value of tidal dissipation of the Earth's rotation (Le Blond and Mysak, 1978; Falkovich, 1992). The amplitude of the condensate can thus grow until the value allowing for wave breaking and front creation. We describe this phenomenon in Sect.3.

Despite the weakness of IG-Rossby interaction, there may be the situations where it is substantial. At Sect.2, we show that if IG waves occupy the region of space larger than cyclone, then the local maximum of IG density appears at the centre of the cyclone. This can decelerate or even stop the westward motion of Rossby waves.

2 Truncated system for IG-Rossby wave interaction

We start from the well known system of shallow-water equations (see e.g. Lesieur, 1990) written for the fluid depth H and the horizontal momentum $\mathbf{p} = H\mathbf{v} =$

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(p, q):

$$\frac{\partial p}{\partial t} - fq + \frac{\partial}{\partial x}\left(\frac{p^2}{H}\right) + \frac{\partial}{\partial y}\left(\frac{pq}{H}\right) + gH\frac{\partial H}{\partial x} = 0 ,$$
$$\frac{\partial q}{\partial t} + fp + \frac{\partial}{\partial y}\left(\frac{q^2}{H}\right) + \frac{\partial}{\partial x}\left(\frac{pq}{H}\right) + gH\frac{\partial H}{\partial y} = 0 ,$$
$$\frac{\partial H}{\partial t} + div\,\mathbf{p} = 0 . \qquad (1)$$

Axis x and y are directed eastward and northward respectively. Coriolis parameter is a function of y: $f = f_0(1+\beta y)$ with β of order of the inverse planet radius R.

Being linearized near the unperturbed state $H = H_0$, $\mathbf{p} = 0$, the system (1) describes two branches of small-amplitude waves. First, there are inertio-gravity waves with the dispersion relation: $\omega_k = \sqrt{f_0^2 + c^2k^2}$. The asymptotic velocity $c = (gH_0)^{1/2}$ is that of gravity waves on a shallow water. This is written neglecting β-term which is possible for the scales of motion $L = 2\pi/k$ less than the so-called intermediate geostrophic radius $R_i = (R\rho^2)^{1/3}$ where the geostrophic (Rossby) radius $\rho = c/f_0$. Second, there are Rossby waves whose frequency is solely due to β-effect (y-dependence of f): $\Omega_k = f\beta k_x/(k^2+\rho^{-2})$. We are interesting in the scales larger than the Rossby radius so that $\Omega \ll \omega$ for any β. We thus come to a classical problem of describing interaction between low-frequency waves and an envelope of high-frequency waves. Our case is quite special, though, with a plenty of unexpected cancellations.

Let us look for the solution of the system (1) in the form

$$H/H_0 = h + h_1 e^{-if_0t} + h_2 e^{-2if_0t} + c.c.$$
$$\mathbf{p} = \mathbf{p}_0 + \mathbf{p}_1 e^{-if_0t} + \mathbf{p}_2 e^{-2if_0t} + c.c.$$

Higher harmonics are irrelevant. Our aim is to obtain nonlinear equations for the amplitude of the first harmonic $\psi = (p_1 + iq_1)/(cH_0)$ describing IG waves and for the slow variable h that describes Rossby waves. The main assumption we make is that the dispersion is small: $L/\rho \gg 1$. We shall keep the terms up to L^{-2} in the expansion. The below calculations are valid even if the nonlinearity parameters $h-1$ (pressure variation) and ψ (Froude number) are of order unity.

Separating in (1) terms with different time exponents, we have in the main order the conditions of the geostrophic balance of the currents and the pressure:

$$p_0 = -\frac{gH_0^2}{2f}\partial_y\left(h^2 + \frac{|\psi|^2}{h}\right) ,$$
$$q_0 = \frac{gH_0^2}{2f}\partial_x\left(h^2 + \frac{|\psi|^2}{h}\right) ,$$
$$h_1 = -\frac{i}{2f_0}(\partial_x - i\partial_y)\psi ,$$
$$q_2 = -ip_2 = -\frac{gH_0^2}{4f_0}(\partial_x - i\partial_y)\frac{\psi^2}{h} .$$

Substituting that into the equations for $\partial\psi/\partial t$ and $\partial h/\partial t$ and neglecting β, after cumbersome calculations we get $\partial h/\partial t = 0$ and

$$2i\frac{\partial\psi}{\partial t} + 2iJ(h,\psi) + h\Delta\psi - \psi\Delta h = 0 . \qquad (2)$$

We use the dimensionless variables f_0t and r/ρ and designate $J(A,B) = \partial_x A\partial_y B - \partial_x B\partial_y A$. The equation (2) has been firstly obtained by Falkovich (1992). Two remarkable cancellations are worth emphasizing: i) there is no high-frequency pressure (like $\nabla|\psi|^2$) in the equation for h without β-effect; ii) the terms cubic in ψ (of order ψ^3/L^2) exactly cancel each other (for any h) in the equation for ψ (note the computational error in the equation (4) in (Falkovich, 1992) which had no influnce on physical conclusions made there). IG self-interaction not only gives no condensate instability or frequency renormalization (Falkovich, 1992) but identically vanishes at this order unlike the linear dispersive term.

Therefore, an actual small dimensionless parameter of the expansion is $|\psi|^2(\rho/L)^2$ so that the nonlinearity parameter $|\psi|$ should be less than L/ρ but can be larger than unity. Physically, this means that the nonlinearity parameter is not v_{osc}/c but v_{osc}/v_{ph} where $v_{ph} = f_0/k$ is the phase velocity of the IG waves and v_{osc} is the oscillatory part of the velocity field.

We thus see that for sufficiently small β (far from the equator) the problem is reduced to studying the behavior of $\psi(x,y,t)$ in a given field $h(x,y)$. This will be discussed in detail elsewhere. Here we answer the question that is of most importance for the destiny of the inverse cascade: can IG waves be trapped by a geostrophic modulations of the atmosphere height? By introducing a new function $\phi = \psi/\sqrt{h}$, we can rewrite the equation (2) as Schrödinger equation for the particle with the mass $1/h$ in the magnetic field $B = -\Delta\ln h$ and in the scalar potential $U = \Delta h/4 - 3(\nabla h)^2/(8h)$. For example, an anticyclone (hump in h) gives the well (negative U) at the center and positive barrier near the edge and vice versa for a cyclone. One can readily found the steady state (with zero energy): $\psi_0(x,y) = h(x,y)$. Since $h \neq 0$ then the ψ-function has no zeroes and thus represents the ground state. At $r \to \infty$ the fluid is unperturbed $h \to 1$ so that our ground state is not a bound state; the function ψ_0 represents the lower boundary of continuous spectrum. Therefore, there are no bound states of ψ in any localized perturbation of h. This generalizes the proof given by Falkovich (1992) for shallow one-dimensional well and confirms the assumption made there that the presence of geostrophic modulations can not prevent the spreading of IG wave packets and stop the inverse cascade.

Still, some nontrivial dynamics can happen if we have multi-scale situations. For example, it has been discovered by Falkovich et al (1994) that if the initial distribution $\psi(x,y,0)$ is much broader than the (cyclonic) well, then IG waves are captured by the well for a long while so that the amplitude in the centre of the cyclon grow at the initial stage of evolution. Only after the time of order $fL_\psi^3/(L_\eta\rho^2)$, the packet spreads. It is thus interesting to study the back influence of ψ on η determined by another small parameter β (which actually is not numerically so small for the Earth atmosphere). Making account of β we get the time variation of the slow variable

$$\frac{\partial h}{\partial t} = \frac{\beta}{2}\frac{\partial}{\partial x}\left(h^2 + \frac{|\psi|^2}{h}\right) . \tag{3}$$

We see that $|\psi|^2$ indeed modifies pressure. According to the nature of Rossby waves, variations of pressure only along x matter. To get a self-consistent system of equations, one should account for β in the equation for ψ as well. We write it for a one-dimensional case:

$$2i\frac{\partial\psi}{\partial t} + h\psi_{xx} - \psi h_{xx} = 2i\beta\psi h_x . \tag{4}$$

The system (3,4) conserves the energy $\int(h^2 + |\psi|^2/h)dxdy$. In the framework of this system, one can readily show that IG condensate ψ =const is stable even at the presence of β-effect.

In the framework of (3,4) one can describe qualitatively the influence of the IG waves on the propagation of geostrophic modulations: In the presence of a broad distribution of IG waves, a maximum of ψ appears at the center of cyclon according to (3). Then one can see from (4) that this diminishes the westward velocity of the cyclon propagation. Such an interaction might take part in maintaining atmospheric blocking. Let us emphasize that this is valid as long as the external source of IG waves is present. Any localized distribution of ψ, h eventually spreads. There is neither steady solutions of soliton type nor collapse events in the weakly nonlinear regime. Note that the weakness of nonlinear coupling thus found explains surprisingly high accuracy of linear computations even at $v/c \simeq 1$ pointed out by McIntyre (1992).

Therefore, the inverse cascade of IG wave turbulence should produce large-scale currents of high amplitude.

3 Wave breaking and front creation

We thus come to the consideration of the strongly nonlinear case with v_{osc}/v_{ph} being arbitrary. We consider large scales so that we neglect dispersion, omitting gravity from the equations (1). We thus assume the typical scales to be much larger than Rossby radius which is possible for the atmospheres of big planets rather than that of the Earth. As far as the Earth atmosphere is concerned, the below analysis can be applied only qualitatively.

The Euler equation now contains only velocity:

$$\partial\mathbf{v}/\partial t + (\mathbf{v}\nabla)\mathbf{v} = \mathbf{v}\times\mathbf{f} . \tag{5}$$

The value H having the meaning of 2d density can be found from continuity equation for the given velocity. The equation (5) can be elementary integrated by characteristics which correponds to the passing to a Lagrangian description. On the characteristic $d\mathbf{r}/dt = \mathbf{v}$, the velocity obeys the Newton equation $d\mathbf{v}/dt = \mathbf{v}\times\mathbf{f}$. Solving an initial value problem (with given $\mathbf{r}(0) = \mathbf{r}_0$ and $\mathbf{v}(x,y,0) = \mathbf{v}_0$) for this equations we get $v_x + iv_y = [v_{0x}(r_0)+iv_{0y}(r_0)]e^{-ift}$. The trajectories of the particles

$$x = x_0 + v_x(r_0)\sin(ft)/f + v_y(r_0)[1-\cos(ft)]/f ,$$
$$y = y_0 + v_x(r_0)[\cos(ft)-1]/f + v_y(r_0)\sin(ft)/f \tag{6}$$

give the solution of (5) in the implicit form. To get explicit solution, one should find $\mathbf{r}_0(\mathbf{r},t)$ from (6) and substitute it into the preceding expression for the velocity. It is clear from that solution that the velocity $\mathbf{v}$ is a periodic function of time with the *constant* frequency f. It means that whatever be the velocity field the nonlinear frequency shift is absent (in the preceding section we have shown that even account of gravity gives no nonlinear frequency renormalization in the long-wave limit).

The general solution (6) enables one to describe the wave breaking. The map $(x_0, y_0) \to (x, y)$ becomes ambiguous when the matrix $A_{ij} = \partial r_i/\partial r_{0j}$ becomes degenerate. Expressing the determinant of A from (6) we get the degeneracy condition:

$$\det A = 1 + f^{-1}\{\mathrm{div}\,\mathbf{v}\sin(ft) + \mathrm{curl}\,\mathbf{v}[1-\cos ft]\} + f^{-2}\left(\frac{\partial v_x}{\partial x_0}\frac{\partial v_y}{\partial y_0} - \frac{\partial v_y}{\partial x_0}\frac{\partial v_x}{\partial y_0}\right) = 0 , \tag{7}$$

where all derivates are taken with respect to r_0 for fixed t.

One could readily find the conditions for breaking from (6). If, say, at $t = 0$ the velocity maximum is U at some circumference of the radius R and there is a particle with the velocity $u > Rf + 2U$ within the circumference, then at $t_0 < 2\pi/f$ that particle will move through the circumference so that trajectory crossing and breaking should occur. The inequality for the velocity is equivalent to the condition $ku/f = u/v_{ph} > 1$ of strong nonlinearity for long waves.

In a general case, the real solution of the equation (7) will appear at some instant of time t_0 as a single point in space. The velocity derivative with respect to the direction parallel to the eigenvector $\mathbf{n}$ of A with zero eigenvalue turns into infinity at this point. This is a wave breaking of the velocity profile. The characteristic velocity amplitude in the breaking time will be of the order of the characteristic phase velocity. The "density" H also turns into infinity at this point which follows from the Lagrangian representation

$$H(\mathbf{r}) = H_0(\mathbf{r}_0)\frac{\partial(x_0, y_0)}{\partial(x, y)} = H_0\det A .$$

Of course, our description is not applicable after the singularity appears. It is reasonable to assume, nevertheless, that a powerful dissipation occurs in the breaking region so that other fluid particles move along characteristics without feeling the influence of that region. The formation of singular region will thus continue by forming the extending line of wave breaking. For time $t - t_0 \ll t_0$, such a process of front creation can be described from (7) by the motion of two branch points $x_0^{(1,2)}$ along the direction transversal to the eigenvector $\mathbf{n}$. Due to the presence of Coriolis force, the breaking line might form spiral-like structures. On the other hand, such fronts will have their proper fine structure and serve as sourses of the generations of short wave desturbances. More sophisticated models of frontogenesis (with the account of pressure, temperature etc) can be found in (Hoskins and Bretherton, 1972). Here we have presented the simplest model which allows for a complete solution.

The wave breaking and front creation thus described in the dispersionless limit is the powerful dissipation mechanism for the long IG waves produced by an inverse cascade in the rotating shallow water.

Let us point out some analogy: a cubic nonlinear term describing self-interaction of electron Langmuir waves in a cold collisionless plasma exactly vanishes in 1d (Zakharov, 1966; Kuznetsov, 1976). The motion of any electron is an oscillation with plasma frequency (Vedenov et al, 1961) and the only mechanism of nonlinear dissipation is the wave breaking if the ions stay at rest (having infinite mass, for instance). In plasma, it is the motion of ions that gives the main nonlinear interaction comparable with dispersion even at small nonlinearity and produces the feedback for the inverse cascade: modulational instability of the Langmuir condensate and wave collapse (Zakharov, 1972). In our problem, there are no particles of the second sort so that nonlinear interaction of long IG waves is significantly suppressed.

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